

A Type of Homogeneity for Continuous Curves

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

CHARLES H. WHEELER, III

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A TYPE OF HOMOGENEITY FOR CONTINUOUS CURVES.¹

By CHARLES H. WHEELER, III.

Introduction. A set of points M is said to be *homogeneous*² if, given any two points x and y of M , it is possible to find a homeomorphism which will send M into itself in such a way that x is sent into y .

A set of points M is said to be *bi-homogeneous*³ if given any two points x and y of M there exists a homeomorphism which sends M into itself in such a way that x is sent into y and y is sent into x .

We will investigate the conditions under which a compact, locally connected continuum may be *cyclic element homogeneous*, i. e., given any two true cyclic elements⁴ of a set M , there exists a homeomorphism which sends M into itself in such a way that one of the given true cyclic elements is sent into the other. A compact locally connected continuum is a continuous curve in the sense that it is a set of points which is the image of the unit interval under a continuous transformation.

The case where M contains only a finite number of true cyclic elements will be completely treated, while some results will be stated for the case where M contains infinitely many true cyclic elements.

Two simple closed curves joined by a simple arc provide an example of a cyclic element homogeneous set. The curve illustrated in Fig. 1 is not cyclic element homogeneous because the true cyclic element marked C_1 cannot be sent into any of the other true cyclic elements by a homeomorphism which sends the set into itself. In Fig. 2, also in Fig. 1, each true cyclic element is homeomorphic with each of the remaining ones, but in Fig. 2 the cyclic element marked C_2 cannot be sent into any of the others by a homeomorphism which sends the set into itself. Fig. 3 is an example of a set of points which contains an infinite number of true cyclic elements and is cyclic element homogeneous.

¹ Received April 18, 1935; revised October 15, 1936.

² See Kuratowski, *Fundamenta Mathematicae*, T. 3 (1922), pp. 14-19, also Mazurkiewicz, *Fundamenta Mathematicae*, T. 5 (1924), pp. 137-146.

³ Kuratowski, *loc. cit.*

⁴ The cyclic elements of a locally connected continuum are (1) all cut points of the continuum and (2) the set of all points conjugate to a point p , where p is any non-cut point of the continuum. A true cyclic element is one which does not reduce to a single point. See G. T. Whyburn, "Concerning the structure of a continuous curve," *American Journal of Mathematics*, vol. 50 (1928), pp. 167-194, and Kuratowski and Whyburn, "Sur les éléments cycliques et leurs applications," *Fundamenta Mathematicae*, T. 16 (1930), pp. 305-331.

1. *Preliminary theorems.* Let M be any compact and locally connected continuum, which we shall consider as a space. Designate by H the smallest A -set⁵ which contains all the true cyclic elements C_i in M , i. e., an A -set containing all the true cyclic elements C_i and not a proper subset of any A -set

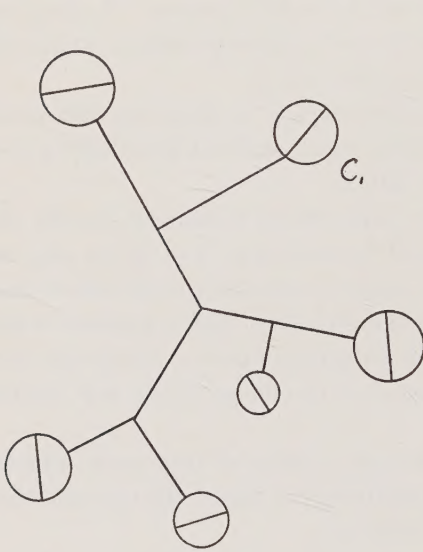


Fig. 1

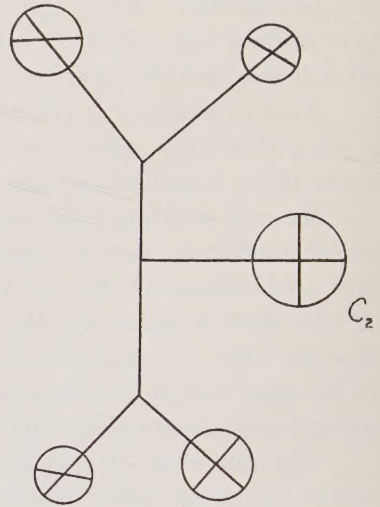


Fig. 2



Fig. 3

containing all the true cyclic elements. The set H may be obtained by taking the product of all A -sets in M which contain all the true cyclic elements. Then since⁶ the product of any number of A -sets is an A -set, it follows H is an

⁵ A closed set which has the property that if $x, y \in A$ then every arc xy in the space is contained in A .

⁶ Cf. Kuratowski and Whyburn, *loc. cit.*, Theorem 4: 1.

A -set and clearly it is the smallest such set containing all the true cyclic elements. If there are only a finite number of true cyclic elements, the set H may be obtained by choosing a non-cut point p_i from each true cyclic element C_i , and forming all the possible pairs of these points. Then for each pair p_i, p_j form the cyclic chain $C_k(p_i, p_j)$. We then have $H = \sum_1^n C_k(p_i, p_j)$.

1. 1. THEOREM. *If M contains only a finite number (> 1) of true cyclic elements, then there exists at least two true cyclic elements each of which has only one point in common with the closure of the remainder of H .*

Proof. This follows immediately from a theorem of G. T. Whyburn.⁷ He proves that if a continuum has more than one cyclic element then it contains at least two nodes, where a node is defined as an end point or a true cyclic element which contains only one cut point. Hence H must contain at least two true cyclic elements which contain only one cut point since it contains no end points.

1. 2. THEOREM. *If T is any homeomorphism which sends M into itself, then $T(H) = H$.*

Proof. The set H is uniquely defined as the smallest A -set containing all the true cyclic elements of M . Any homeomorphism T will send H into an A -set which contains all the true cyclic elements of M , thus $H \subset T(H)$.

Since T^{-1} is a homeomorphism, $H \subset T^{-1}(H)$. Operating upon this with T we have $T(H) \subset TT^{-1}(H) = H$. Thus $T(H) \subset H$. It follows from the above two inclusions that $T(H) = H$, and the theorem is proved.

We will consider H to be our space for the remainder of this paper. H is a compact locally connected continuum.

1. 3. *Definition.* A set H is said to be *cyclic element homogeneous* if, given any two true cyclic elements of H , then there exists a homeomorphism T such that $T(H) = H$ and one of the given true cyclic elements is sent into the other.

From this definition we have immediately the following:

1. 4. THEOREM. *If H is cyclic element homogeneous, then each true cyclic element contains the same number (finite or infinite) of cut points of H .*

2. *The case where H contains only a finite number of true cyclic elements.* If the set H contains only a finite number (> 1) of true cyclic elements,

⁷ "Concerning the structure of a continuous curve," *loc. cit.*, p. 180.

there exists a true cyclic element C_s which contains only one point which cuts H by 1. 1. Then every true cyclic element of H has only one point in common with the closure of the remainder of H , by 1. 3, i. e., $C_i \cdot \overline{H - C_i} =$ a single point.

Definition. Let p_1^i be the point of C_i such that $p_1^i \in \overline{H - C_i}$. The sum of all true cyclic elements which contain p_1^i is called a *cluster* of true cyclic elements. The point $p_1^i = p_j$ is called the center of the cluster K_j .

Then $K_j = \sum_{i=1}^n C_i^j$ and $\prod_{i=1}^n C_i^j = p_j$. The centers of every two clusters are joined by a simple arc in H , since H is an A -set. The arc $p_j p_k$ is such that

$$p_j p_k \cdot K_j = p_j \quad \text{and} \quad p_j p_k \cdot K_k = p_k,$$

for we have seen above that no C_i^j of K_j can have more than one point in common with $\overline{H - C_i^j}$.

2. 1. THEOREM. *If H is cyclic element homogeneous and contains only a finite number of true cyclic elements, then there is the same number of true cyclic elements in each cluster.*

Proof. Suppose the contrary, then there exists some two clusters such that

$$K_1 = \sum_1^n C_i^1, \quad K_2 = \sum_1^m C_i^2, \quad n > m.$$

The points p_1 and p_2 are the centers of the clusters K_1 and K_2 respectively. Since H is cyclic element homogeneous, let $T(H) = H$ in such a way that $T(C_1^1) = C_1^2$. Now $C_1^1 \cdot \overline{K_1 - C_1^1} = p_1$ and $C_1^2 \cdot \overline{K_2 - C_1^2} = p_2$, and hence $T(p_1) = p_2$.

Since $C_2^1 \supset p_1$, then $T(C_2^1) =$ some C_i^2 for some i , $i = 2, 3, \dots, m$. This is also true for the remaining $n - 2$ true cyclic elements in K_1 , but there are $m - n$ less true cyclic elements in K_2 than in K_1 , so this is impossible. Thus the supposition that the clusters do not contain the same number of true cyclic elements leads to a contradiction, and the theorem is proved.

2. 2. COROLLARY. *If a true cyclic element of one cluster is transformed by a homeomorphism T into a true cyclic element of another cluster, then the first cluster is transformed into the second cluster by T .*

2. 3. THEOREM. *If H is cyclic element homogeneous and contains only a finite number of true cyclic elements, then no cluster cuts H .*

Proof. Suppose there exists no cluster which does not cut H . There are only a finite number of clusters since there are only a finite number of true cyclic elements in H . Then $H - K_1 = H_1^1 + H_2^1$, where

$$H_1^1 \cdot \overline{H_2^1} = \overline{H_1^1} \cdot H_2^1 = 0 \quad \text{and} \quad H_1^1 \neq 0 \neq H_2^1.$$

H_1^1 contains at least one cluster K_{i_1} . Let K_{n_2} be the cluster in H_1^1 with the least subscript. Then $H - K_{n_2} = H_1^2 + H_2^2$, where

$$H_1^2 \cdot \overline{H_2^2} = \overline{H_1^2} \cdot H_2^2 = 0, \quad H_1^2 \neq 0 \neq H_2^2 \quad \text{and} \quad H_2^2 \supset K_1.$$

H_1^2 contains at least one cluster K_{i_2} , $K_{i_2} \neq K_{n_2}$. Let K_{n_3} be the cluster in H_1^2 with the least subscript. Then $H - K_{n_3} = H_1^3 + H_2^3$, where

$$H_1^3 \cdot \overline{H_2^3} = \overline{H_1^3} \cdot H_2^3 = 0, \quad H_1^3 \neq 0 \neq H_2^3 \quad \text{and} \quad H_2^3 \supset K_{n_2}.$$

H_1^3 contains at least one cluster K_{i_3} , $K_{i_3} \neq K_{n_3}$.

This can be carried on indefinitely, but this is impossible for there are only a finite number of clusters. Thus there exists at least one cluster K_t which does not cut H .

Let K_p be any cluster of H different from K_t . By 2.2 $H - K_t$ is homeomorphic with $H - K_p$ and hence K_p can not cut H . Thus no cluster cuts H .

2.4. Definitions.

2.41. A compact locally connected continuum H is said to be *symmetrical* with respect to a cyclic element C (whether C be a true cyclic element or not) if every component of $H - C$ is homeomorphic with every other component of $H - C$. We will call C the center of symmetry.

2.42. A set of points H is said to be *cyclic symmetrical* with respect to a cyclic element C (whether C be a true cyclic element or not) if it is symmetrical and any true cyclic element of one component of $H - C$ may be sent into any true cyclic element of another component of $H - C$ by a homeomorphism which sends the first component into the second.

2.43. A *major branch* at a branch point x is the component of $H - x$ which contains the center of symmetry.

2.44. A *minor branch* at a branch point x is a component of $H - x$ which does not contain the center of symmetry.

2.5. THEOREM. *If H is cyclic element homogeneous and contains only a finite number of true cyclic elements, then there exists a point c such that H is cyclic symmetrical with respect to c .*

Proof. We saw that the centers of every pair of clusters are joined by an arc in H , and that this arc has only the centers of the clusters in common with the clusters. Also by 2.3 no cluster cuts H . Thus $H - \sum_1^m K_i$ is an acyclic curve with a finite number of branches.

There is a center p_i of a cluster K_i at the end of each branch. Form all the possible pairs of the points p_i . Take a pair p_1, p_2 which has a maximum number of branch points on the arc joining them. If this number is odd let the middle branch point be c , if it is even let c be any point on the open arc joining the two middle branch points. The point c is uniquely determined in the case where the maximum number is odd, and uniquely determined to the extent of being any one of the points on an open arc in the case where the maximum number is even. Now

$$H - c = S_1 + S_2 + S_3 + \cdots + S_n,$$

where $S_1, S_2, \cdots, S_n$ are components. If the number of branch points on the arc joining p_1 and p_2 is even there are only two components, while if this number is odd there may be any finite number of components. Let $S_1 \supset p_1$ and $S_2 \supset p_2$.

Number the branch points on S_1 in the following way: Let the first branch point out from c be x^1 , then on each minor branch from x^1 let the first branch points be $x_1^2, x_2^2, \cdots, x_m^2$. The branch point on the branch containing p_1 we will call x^2 . Let the first branch points on the minor branches from x_1^2 be $x_{1,1}^3, x_{1,2}^3, \cdots, x_{1,k}^3$. The branch point on the branch containing p_1 we will call x^3 . Continue in this manner until all the branch points have been numbered. Number the branch points on S_2 in the same manner except with the use of y instead of x .

The number of branch points from c to p_1 is the same as the number from c to p_2 ; let us assume this number is k .

We must now show that there exists a homeomorphism T such that $T(S_1) = S_2$. Since H is cyclic element homogeneous, let T be a homeomorphism of H into itself such that $T(C_1^1) = C_1^2$ where $C_1^1 \supset p_1$ and $C_1^2 \supset p_2$. Then $T(p_1) = p_2$ and $T(K_1) = K_2$. The arc $p_1 x^k$ is sent into the arc $p_2 y^k$ and x^k into y^k by T . At the branch points x^k and y^k there are the same number of branches because x^k and y^k correspond to each other under a homeomorphism. There is no branch point on any of the minor branches at x^k or y^k for if there were one on some minor branch, say at y^k , there would be more branch points on the arc joining one of the centers p_i (on this branch) and p_1 than on the arc $p_1 p_2$. Then p_1, p_2 would not be a maximal pair.

The arc $x^k x^{k-1}$ is sent into the arc $y^k y^{k-1}$ by T so that x^{k-1} is sent into y^{k-1} . There are the same number of branches at x^{k-1} as at y^{k-1} because x^{k-1} and y^{k-1} correspond to each other under a homeomorphism. There can not be more than one branch point on any of the minor branches at x^{k-1} or y^{k-1} ; for if there were, p_1, p_2 would not have been a maximal pair. There must be one branch point on each of the minor branches at x^{k-1} and y^{k-1} and the number of minor branches at these branch points is the same as the number of minor branches at x^k or y^k .

Now continue down the major branch at x^{k-1} and y^{k-1} . The arc $x^{k-1} x^{k-2}$ is sent into the arc $y^{k-1} y^{k-2}$ by T , and then x^{k-2} is sent into y^{k-2} . The number of minor branches at x^{k-2} and y^{k-2} is the same, because x^{k-2} and y^{k-2} correspond to each other under a homeomorphism. The minor branches at x^{k-2} are homeomorphic with the minor branches at y^{k-2} . On continuing in this manner until we reach $T(x^1) = y^1$, it can be shown that the minor branches at x^1 are homeomorphic with the minor branches at y^1 . Then the arc $x^1 c - c$ is sent into the arc $y^1 c - c$ by T . Therefore $T(S_1) = S_2$.

If there are only two components of $H - c$, the theorem is proved. In the case where there are more than two components of $H - c$, take any component S_3 different from S_1 and S_2 . Let p_3 be the center of the cluster K_3 such that the number of branch points on the arc joining it to c is a maximum for all centers p_i in S_3 . Let the number of branch points on this arc be m . Then $m \leq k$, for if it were greater p_1, p_2 would not have been a maximal pair. Number the branch points on S_3 the same as on S_1 , denoting them by z instead of x .

Since H is cyclic element homogeneous, let $T(H) = H$ in such a way that $T(C_1^3) = C_1^3$, where $C_1^3 \subset K_1 \subset S_1$ and $C_1^3 \subset K_3 \subset S_3$. Then $T(K_1) = K_3$, $T(p_1) = p_3$, $T(p_1 x^k) = p_3 z^m$ and $T(x^k) = z^m$. The number of branches at x^k is the same as the number at z^m , for x^k and z^m correspond to each other under a homeomorphism. There is no branch point on any of the minor branches at x^k or z^m , for if there were, p_3 would not have been a maximal number of branch points from c . Then $T(x^k x^{k-1}) = z^m z^{m-1}$ and $T(x^{k-1}) = z^{m-1}$. The number of branch points at x^{k-1} and z^{m-1} are the same since x^{k-1} and z^{m-1} correspond under a homeomorphism. The minor branches at z^{m-1} are homeomorphic and homeomorphic to the minor branches at x^{k-1} .

It is thus seen that if $m = k$, S_1 is homeomorphic with S_3 . If $m < k$, say $m + 1 = k$, then one minor branch at x^1 is sent into S_3 and x^1 is sent into c . Since x^1 and c correspond to each other under a homeomorphism, there are the same number of branches at x^1 as there are at c . This may or may not be true; if not, it is seen immediately that $T(H) \neq H$, and therefore

$m = k$. If the number of branches at x^1 and c are the same, consider the number of branch points on the arc from x^1 to $p_2 \in S_2$. There are $k + 1$. The branch from x^1 containing these must be sent into a component of $H - c$ different from S_3 ; but we have seen that the maximum number of branch points from c to the center of any cluster in $H - c$ was k . Therefore this is impossible and $T(H) \neq H$. Thus $m = k$. Therefore all of the components of $H - c$ are homeomorphic.

It must now be shown that any true cyclic element in one component may be sent into any true cyclic element in another component by a homeomorphism which sends the first component into the second. Let C_1^i be any true cyclic element in S_r and C_1^j any in S_t . Now $C_1^i \subset K_i$ and $C_1^j \subset K_j$. Since H is cyclic element homogeneous, let $T(H) = H$ in such a manner that $T(C_1^i) = C_1^j$. It has already been shown that S_r and S_t are homeomorphic, that every cluster on S_r and S_t are the same number (k) of branch points away from c , that at each i -th branch point out from c ($i = 1, 2, 3, \dots, k$) there are the same number of minor branches which are homeomorphic with one another. Let the branch points on S_r and S_t be numbered in the same manner as were the branch points on S_1 and denoted by u and v , respectively. We have then $T(K_i) = K_j$, $T(p_i) = p_j$, $T(p_i u^k) = p_j v^k$ and $T(u^k) = v^k$. The remaining minor branches at u^k are sent into the remaining minor branches at v^k by T . Then $T(u^k u^{k-1}) = v^k v^{k-1}$ and $T(u^{k-1}) = v^{k-1}$. The remaining minor branches at u^{k-1} are sent into the remaining minor branches at v^{k-1} by T . Continuing this finally yields $T(u^1) = v^1$. The remaining minor branches at u^1 are sent into the remaining minor branches at v^1 . Then $T(u^1 c - c) = v^1 c - c$, thus $T(S_r) = S_t$. Q. E. D.

2.6. *Summary.* We have proved that if H is cyclic element homogeneous and contains only a finite number of true cyclic elements C_i then

- 1) $C_i \cdot \overline{H - C_i} =$ a single point, for each i .
- 2) Each cluster K_i has the same number of true cyclic elements.
- 3) $\overline{H - K_i}$, for every i , is connected.
- 4) $H - \sum_1^n K_i$ is an acyclic curve with a finite number of branches.
- 5) There exists a point c such that
 - a. Every cluster is the same number of branch points distant from c .
 - b. At each i -th branch point from c , ($i = 1, 2, \dots, k$) there are the same number of branches, and all the minor branches are homeomorphic.
 - c. On every component of $H - c$ there are the same number of branch points and the same number of clusters.

- d. Every component of $H - c$ is homeomorphic with every other and any true cyclic element of one component may be sent into any true cyclic element of any other component by a homeomorphism T which sends the first component into the second and which sends H into itself.
- 6) All the true cyclic elements are homeomorphic, and if $p_i^r \in C_r$ and $p_i^s \in C_s$ such that $C_r \cdot \overline{H - C_r} \supset p_i^r$ and $C_s \cdot \overline{H - C_s} \supset p_i^s$, then the homeomorphism $W(C_r) = C_s$ is such that $W(p_i^r) = p_i^s$ for some i .

Fig. 4 is an example of a set L which is cyclic element homogeneous. The point c is the center of symmetry, and there are three components of $L - c$.

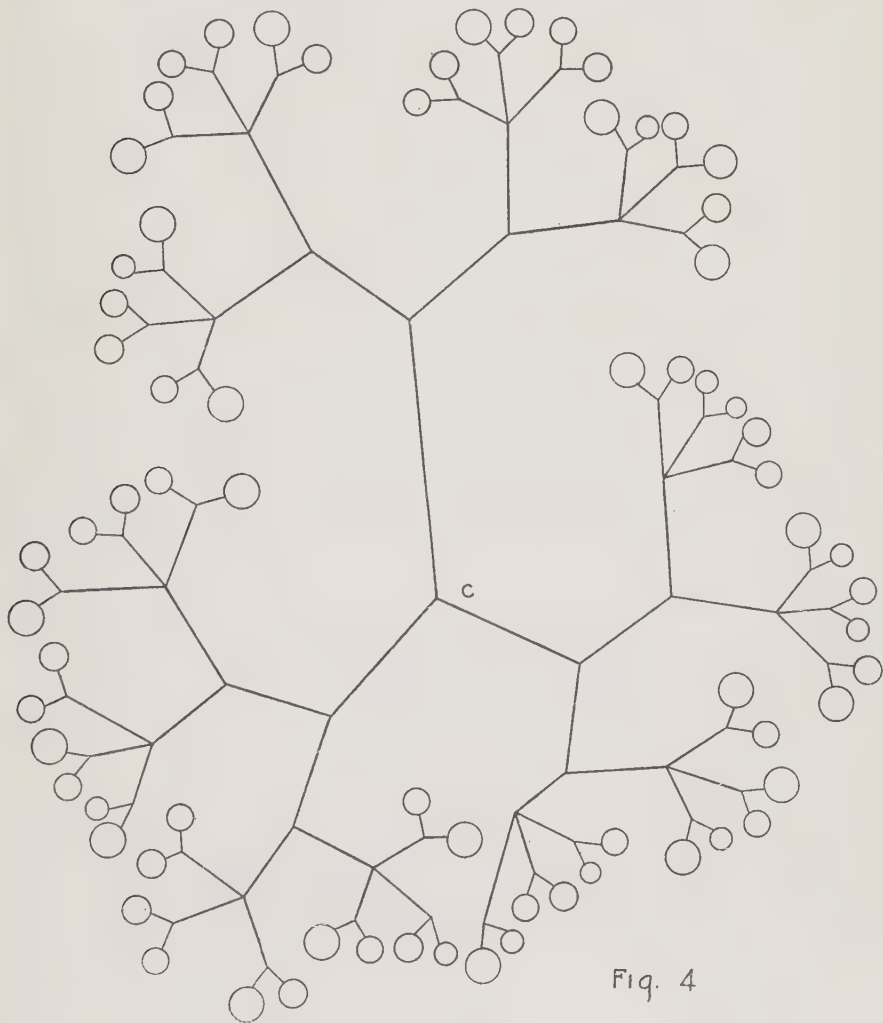


Fig. 4

2.7. THEOREM. *If H contains only a finite number of true cyclic elements, in order that H be cyclic element homogeneous it is necessary and sufficient that (1) the true cyclic elements be grouped in clusters with the same number in each cluster, (2) no cluster cuts H , (3) there exists a point c of H such that each cluster is the same number, k , of branch points away from c , (4) at each i -th branch point ($i = 1, 2, \dots, k$) from c there are the same number of branches, and (5) if C_r and C_s are any two true cyclic elements of H , there exists a homeomorphism of C_r into C_s which sends the cut points of H on C_r into the cut points of H on C_s .*

Proof. The necessity follows from 2.5. To show the sufficiency take any two components S_1, S_2 of $H - c$. Denote the branch points on S_1 by x and those on S_2 by y , as was done in 2.5. Go out along the arcs from c to x^1 and to y^1 lying in S_1 and S_2 respectively. There are the same number of branches at x^1 as at y^1 , by hypothesis. Out from x^1 and y^1 on each minor branch there is a second branch point from c . There are the same number of branches at each of these by hypothesis. Out from the second branch points from c on each of the minor branches is the third branch point from c , and there are the same number of branches at each of these.

Continue this until we get to the k -th branch points from c . By hypothesis there are the same number of branches at each of these points. Out from the k -th branch points on each of the minor branches is a cluster of true cyclic elements, for otherwise this branch would not have been in H . There can not be any branch point on any of these branches, for if there were there would be at least two clusters which had more than k branch points on the arc from the center of the cluster to c . There is no cluster on any minor branch from x^1 and y^1 before the k -th branch point, for if there were there would be less than k branch points on the arc from its center to c . Thus there is a cluster of true cyclic elements at the end of each minor branch from the k -th branch points. By hypothesis there is the same number of true cyclic elements in each cluster. Since no cluster cuts H , every two of the clusters are joined by an arc in H which passes through at least one branch point. Each true cyclic element contains only one point which cuts H , namely the center of the cluster in which the true cyclic element is contained. This follows from the fact that no cluster cuts H .

It must now be shown that there exists a homeomorphism T such that if there be given any two true cyclic elements C_i, C_j of H , then $T(H) = H$ in such a way that $T(C_i) = C_j$. Take any two true cyclic elements C_1 and C_2 . By hypothesis there exists a homeomorphism W such that $W(C_1) = C_2$ and $W(p_1) = p_2$ where $p_1 \in C_1 \cdot \overline{H - C_1}$ and $p_2 \in C_2 \cdot \overline{H - C_2}$. Define the homeo-

morphism $T = W$ over C_1 . There are the same number of true cyclic elements in each cluster, hence the definition of T can be extended so that $T(K_1) = K_2$ where $K_1 \supset C_1$ and $K_2 \supset C_2$. If p_1x^k and p_2y^k are the arcs from K_1 and K_2 to the branch points x^k and y^k respectively, we so extend the definition of T that $T(p_1x^k) = p_2y^k$, and $T(x^k) = y^k$; also $T(b^i_{x,k}) = b^i_{y,k}$ where $b^i_{x,k}$ are the minor branches at x^k which do not contain p_1 and $b^i_{y,k}$ the minor branches at y^k which do not contain p_2 , $i = 2, 3, \dots, m$. If x^kx^{k-1} and y^ky^{k-1} are the arcs from x^k and y^k to the $(k-1)$ -st branch points, we define $T(x^kx^{k-1}) = y^ky^{k-1}$ and $T(x^{k-1}) = y^{k-1}$; also $T(b^i_{x,k-1}) = b^i_{y,k-1}$ where $b^i_{x,k-1}$ are the minor branches at x^{k-1} not containing x^k and $b^i_{y,k-1}$ the minor branches at y^{k-1} not containing y^k , $i = 2, 3, \dots, j$. Continue in this manner until $x^j = y^j$ or until c is reached on both components. If b^1 and b^2 are the branches from x^j or c , as the case may be, which contain C_1 and C_2 respectively, for each point $x \in b^2$, define $T(x) = T^{-1}(x)$; for each point $x \in H - b^1 - b^2$, define $T(x) = x$. We thus have a homeomorphism T which sends C_1 into C_2 , the component of $H - c$ which contains C_1 into the component of $H - c$ which contains C_2 , and H into itself.

2.8. *Definition.* A set of points H is said to be *bi-cyclic element homogeneous* if, given any two true cyclic elements of H , there exists a homeomorphism T such that $T(H) = H$ and the true cyclic elements are sent into each other.

From the way the homeomorphism T was defined in 2.7, it is seen that if a set H satisfies the conditions of 2.7 it is bi-cyclic element homogeneous.

As was stated in the introduction, Fig. 3 is an example of a space M which is cyclic element homogeneous but it is not bi-cyclic element homogeneous. The true cyclic element C_1 may be sent into any other true cyclic element of the space by a homeomorphism which sends M into itself; but C_1 can not be sent into C_2 by a homeomorphism which sends C_2 into C_1 and M into itself.

It may be remarked that the finite case just treated could have been reduced to the consideration of an acyclic curve which was end point homogeneous. However, little if any advantage in simplicity seems to accrue from such a reduction.

3. *The case of infinitely many true cyclic elements.* Although no complete solution for this case of the problem has yet been obtained, we shall state here some results bearing on certain important phases of it.

3.1. If H is cyclic element homogeneous and contains infinitely many true cyclic elements which are grouped in clusters, where no cluster cuts H ,

then there are a finite number of clusters with an infinite number of true cyclic elements in each cluster. Also under this hypothesis there exists a point c such that each cluster is the same number of branch points away from c and at each i -th branch point ($i = 1, 2, \dots, k$) from c there are the same number of branches. It can be shown that the necessary and sufficient condition for this case is similar to that of the finite case.

3.2. Let H satisfy the conditions: (1) it contains infinitely many true cyclic elements $\{C_j\}$, (2) each cut point is contained in exactly two true cyclic elements, (3) each component of $H - C_j$ has a different boundary point, (4) the set of cut points of H is totally disconnected, and (5) no point p is the limit of true cyclic elements in more than one component of $H - p$.

Under these restrictions, a necessary condition that H be cyclic element homogeneous is that when C_s and C_t are any two true cyclic elements of H , there exists a homeomorphism of C_s into C_t such that one particular cut point in C_s is sent into one particular cut point in C_t and the remaining cut points in C_s are sent into the remaining cut points in C_t .

Thus far it has not been shown that this condition is sufficient for H to be cyclic element homogeneous. But a sufficient condition is that when C_s and C_t are any two true cyclic elements of H , there exists a homeomorphism T of C_s into C_t such that two particular cut points in C_s are sent into two particular cut points in C_t and the remaining cut points in C_s are sent into the remaining cut points in C_t .

It is to be noted that the sufficient condition is stronger than the necessary condition.

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